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CHAPTER 9

Chaos Theory and Rationality in Economics

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A central assumption in economic theory is that of rational behavior by economic agents, although this has been widely criticized by many noneconomists. Since the work of Muth (1961), this assumption has increasingly taken the form of rational expectations, that agents over time on average accurately predict the future. This somewhat simplistic formulation of the assumption depends on agents knowing the underlying "true" model of reality and fully utilizing information that, in turn, is assumed to arrive in a random pattern that reflects a normal distribution. Agents may make errors from time to time, but these errors tend to cancel each other out over time.[1]

Muth (1961) originally applied the theory to explain behavior in agricultural markets. Its use has since spread throughout economics at both the microeconomic and macroeconomic levels, as well as into financial theory. Among the most controversial applications have been in macroeconomics (Lucas and Sargent 1981), where models using the assumption have been used to support a "New Classical" view that systematic stabilization policies by governments will be ineffective and can only cause inflation in the long run. The essential argument is that economic agents will rationally forecast the effects of government policies and will act in ways that nullify those effects.

The assumption has acquired the status of a Lakatosian *hard core* axiom of the economic research program (Rosser 1993), despite some empirical findings that it does not hold for a variety of economic agents (Lovell 1986). The argument for this hard core axiom view has been articulated by Sargent (1982, 382), who has asserted that it is a hypothesis not amenable to empirical testing because of "the logical structure of rational expectations as a modeling strategy, the questions that it invites researchers to face, and the standards that it imposes for acceptable answers to those questions."

A significant complication for the use of this assumption has arisen in the form of the increasing awareness of the widespread reality of nonlinearity in many dynamical economic systems. Such nonlinearity can lead to chaotic dynamics in economic systems.[2] A number of theoretical models have been developed in different areas of economics that assume rational expectations but can generate chaotic dynamics for certain parameter values.

This is ironic because in a situation of chaotic dynamics, the condition of sensitive dependence on initial conditions holds. This implies that even a minute error in estimation can lead to serious errors in long-run forecasting. This raises very serious doubts about the realistic applicability of the rational expectations assumption under such circumstances. It is this contradictory situation that is the central theme of this chapter.

Chaotic Microeconomic Models with Rational Expectations

Chaotic Preference Cycles

Generally, in microeconomic analysis it is assumed that individual preferences are constant and that a person's behavior reflects their responses to changes in relative prices or income. However, it has been recognized since at least Pareto (1909) that this is a highly unrealistic assumption, and some economists have occasionally attempted to develop models that dynamicize preferences. Such an effort that is both consistent with rational expectations and also allows for chaotic patterns of individual consumption behavior is due to Benhabib and Day (1981).

They propose that a person's preferences reflect past consumption patterns and that such relationships may generate cyclical patterns of preferences and consumption as between two goods. Thus people may alternate between ski and beach holidays. Letting the goods be x and y, with m the person's money, they assume that the person's utility from consuming the goods can be represented by the standard Cobb-Douglas utility function,

$$U(x,y) = x^a y^{1-a}. \tag{1}$$

They hypothesize two different dynamic processes that both operate through changing the value of a over time. The first one, which is analogous to the Lorenz (1963) climate model, is given by

$$a_{t+1} = \alpha x_t y_t, \tag{2}$$

which, in turn, implies a dynamic demand curve

$$x_{t+1} = \alpha m x_t(m - x_t). \tag{3}$$

The control parameter for this process is αm^2. At values slightly greater than one there is a stationary demand curve. As its value increases, period-doubling bifurcations emerge with the resulting emergence of cyclical consumption behavior. Chaotic dynamics appear for values greater than 3.57.

The second process is given by

$$a_{t+1} = x_t e^{\alpha(1-x_t)}, \tag{4}$$

which implies a dynamic demand function

$$x_{t+1} = m x_t e^{\alpha(1-x_t)} \tag{5}$$

If $m = 1$, this is analogous to a population dynamics model due to May and Oster (1976) for which α is the control parameter. Chaos appears when it approximately equals 2.6924.

Both of these processes imply that as α declines, higher values of m are required for chaotic dynamics to occur. Given that α is the *experience dependent parameter* and m is income, Benhabib and Day (1981, 463) conclude that "the two models then characterize experience dependent demand as converging to a stable long run pattern for relatively low incomes, but exhibit increasing instability and eventually become completely erratic as income grows reflecting the whimsical, seemingly arbitrary behavior of the complacent, or the very rich!"

Interpersonal Utility Effects and Chaotic Dynamics

Another realistic but infrequently allowed aspect of individual preferences in microeconomic analysis is the sociological observation that a person's preferences may depend on the behavior of others, an observation originally made by Veblen (1899) and later studied by Leibenstein (1950). In particular the latter distinguished between *bandwagon* effects, wherein one wants to buy something when others are buying it, and *snob* (or reverse bandwagon) effects wherein one does *not* want to buy something when others are buying it.

Granovetter and Soong (1986) postulate that a person could exhibit both effects with respect to a particular good for different levels of market saturation. Thus someone might be inclined to buy a push-button phone when 20 percent of their friends do so, for bandwagon reasons, but may incline toward an old rotary phone when 90 percent of their friends have push-button ones, for snob reasons. Such a situation implies a nonlinear dynamic demand process and if it is sufficiently nonlinear the dynamics can be chaotic.[3] Figure 9.1, taken from Granovetter and Soong 1986, shows the demand curve in such a situation. As price, P, declines below P_1, period-doubling bifurcations appear. P^* is the threshold of chaotic dynamics.

Iannacone (1989) has criticized this model for having consumer preferences depend only on the behavior of others during the most recent period. He

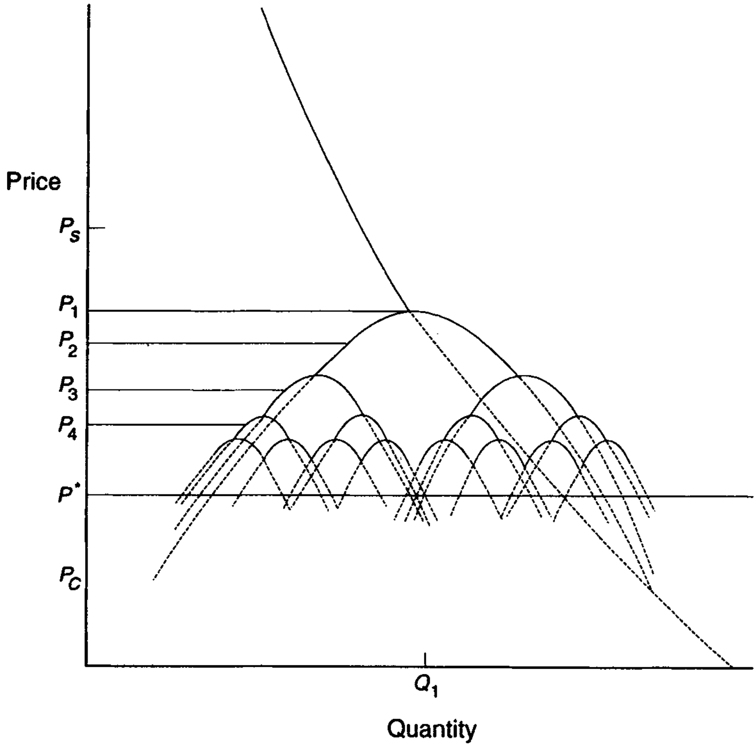

Fig. 9.1. Chaotic dynamic demand process. (From Granovetter and Soong 1986. Reprinted with permission.)

shows that an adaptive expectations strategy drawn from Heiner 1989 wherein consumers respond to a weighted average of past behavior by others reduces the likelihood of chaotic dynamics. This is an argument we shall consider in more detail later.

Chaotic Cobwebs

Originally conceived by Cheysson (1887), cobwebs are oscillatory and possibly explosive dynamics in markets that can arise when there are lags in the production process, the classic example being in agriculture. Ironically, the original motivation behind Muth's (1961) development of the rational expectations assumption was to rule out such outcomes for agricultural markets. He argued that farmers would rationally forecast the true long-run equilibrium and would not foolishly indulge in alternating increases and decreases in output in response to short-run price fluctuations.

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However, it has since been shown that oscillatory and even chaotic dynamics can emerge because of production lags if supply or demand curves are nonlinear and nonmonotonic, even with rational expectations. Artstein (1983) posited partially downward-sloping supply curves for agricultural commodities if farmers (rationally) forecast that very low prices in one year will trigger government price supports in the next year. Artstein shows that such a case can generate a three-period cycle, thus implying chaotic dynamics. Jensen and Urban (1984) derived similar results for models with backward-bending supply curves (not uncommon in labor markets and fisheries) and double-valued demand curves (much rarer, perhaps Irish potatoes in the mid–nineteenth century).

Chiarella (1988) showed the possibility of chaotic cobwebs even with normally shaped supply and demand curves. However, his results are not consistent with rational expectations because they depend on the existence of a lag in expectations formation as well as a lag in production, realistic as that may be.[4] This model of Chiarella's underlies the empirical studies of the pork and milk cycles by Chavas and Holt (1991, 1993).

Chaotic Macroeconomic Models with Rational Expectations

Overlapping Generations Models

The first macroeconomic models to exhibit chaotic dynamics that are consistent with rational expectations (Benhabib and Day 1980, 1982),[5] labeled "Strong New Keynesian" by Rosser (1990), depend on the use of the overlapping generations concept developed by Allais (1947) and Samuelson (1958). In each period of time let there be two generations alive, with the young possessing endowments of w_y and the old possessing endowments of w_o and the old possessing fiat money at the beginning of time equal to M. Let $p(t)$, $c_y(t)$, and $c_o(t)$ be, respectively, prices, consumption by the young, and consumption by the old in time t.

The young seek to maximize utility

$$\max U[c_y(t), c_o(t+1)], \tag{6}$$

subject to their intertemporal budget constraint

$$p(t)c_y(t) + p(t+1)c_o(t+1) = p(t+1)w_o. \tag{7}$$

These generate an intergenerational offer curve, O, by the young as $p(t)$ and $p(t+1)$ vary. The old in time 1 face the constraint

$$p(1)c_o(1) = p(1)w_o + M. \tag{8}$$

Given a Ricardian intertemporal production possibilities frontier,

$$R = [(c_y, c_o) : c_y + c_o = w_y + w_o], \tag{9}$$

a set of perfect foresight (hence rational expectations) equilibria are given by a sequence of prices and consumption levels satisfying equations 6, 7, and 8.

The pattern generated by this sequence will depend on the "humpiness" of the intergenerational offer curve, O. Sufficient humpiness can lead not merely to endogenous cycles, but to chaotic cycles such as that shown in figure 9.2. Benhabib and Day (1982) show that this humpiness will depend positively on the variability of the constrained marginal rate of substitution between individuals' present consumption and future consumption. The precise sufficiency conditions for chaos also depend on the endowments of the young and on the rate of population growth. They also show that these chaotic trajectories are efficient, that is, Pareto-optimal.[6]

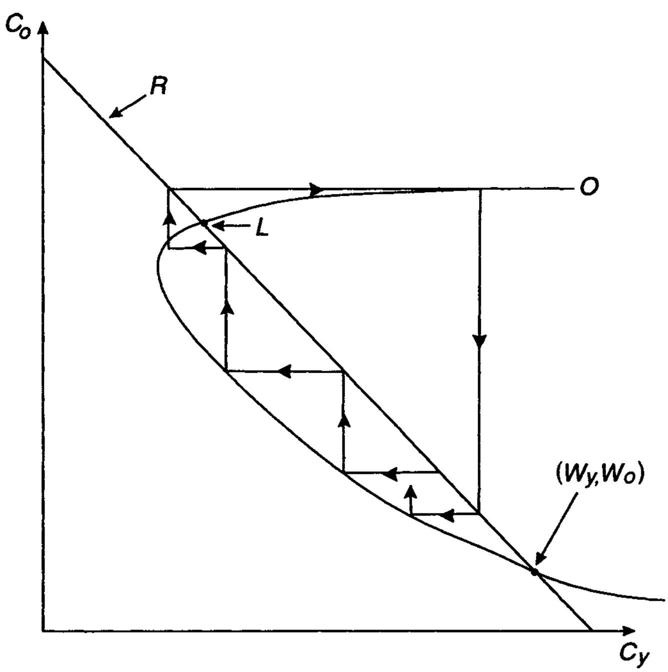

Fig. 9.2. Chaotic intergenerational offer curve

A somewhat different overlapping generations perfect foresight model that can generate endogenous chaotic dynamics is due to Grandmont (1985). His model focuses on interest rate changes initiating conflicts between intertemporal wealth effects and intertemporal substitution effects. If older agents have a marginal propensity to consume leisure sufficiently greater than that of young agents, then chaotic dynamics can emerge. More specifically, let a be real wealth and V the indirect utility function of either generation. Then, the Arrow-Pratt relative degree of risk aversion, which measures the curvature of V, is

$$R(a) = -V''(a)a/V'(a). \tag{10}$$

Chaotic cycles will occur if $R_o(a_o) > R_y(a_y)$.[7]

Figure 9.3 illustrates such a case derived from Grandmont's (1985) model. The axes are real money balances in time t, μ, on the horizontal and the perfect foresight dynamics of real money balances in time $t + 1$, $x(\mu)$, on

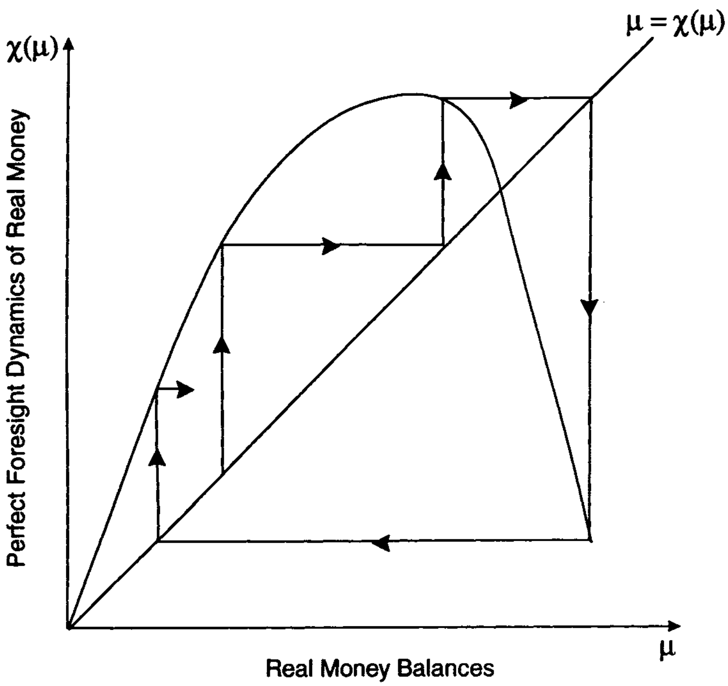

Fig. 9.3. Perfect foresight model in overlapping generations with endogenous chaotic dynamics. (From Grandmont 1985.)

the vertical. The humpiness of $x(\mu)$ increases with $R_o(a_o)$. Grandmont argues that Keynesian-style governmental policies of proportional intergenerational money transfers can pin down expectations about the interest rate and eliminate any cyclical outcomes. However, this depends on both the government being able to make accurate forecasts under conditions of chaotic dynamics with the attendant problem of sensitive dependence on initial conditions and people having complete confidence in government policies.

Brock (1988a) has dismissed the realism of such assumptions as amounting to a "nirvana fallacy." Indeed, Dwyer (1992) has developed a model wherein stabilization policy generates chaotic dynamics in a situation when otherwise such would not occur. This echoes monetarist critiques of discretionary Keynesian policies as generating business cycles when otherwise they would not occur. Such an argument has been reinforced by DeCoster and Mitchell (1992), who show that a chaotic policy will be magnified into even more chaotic dynamics, even when the economy is basically nonchaotic and reflects linear rational expectations.

Discounting and Chaos in Infinite Horizon Models

Strong advocates of the rational expectations approach object to the kinds of models presented in the previous section on the grounds that they either do not involve optimization over an infinite time horizon (Benhabib and Day 1982; Grandmont 1985) or involve some kind of incompleteness of markets (Woodford 1989). Thus it is argued that they do not represent truly complete general equilibrium solutions. Defenders of the above approaches may cite their apparently greater realism, but the critics note that agents may well have time horizons beyond personal lifetimes, either for family bequest motives (Barro 1974) or because important agents may be transpersonal entities such as corporations or governments with presumably very long time horizons. Thus the question arises as to whether or not chaotic dynamics can arise in such models with completely general infinite horizon equilibria. The answer is yes.

One way that chaos can appear in such models is if there is a failure to converge to a steady-state equilibrium, even though one exists. For optimal, infinite horizon, multisector models, Sutherland (1970) initially showed that a low discount factor (high real interest rate) could lead to such nonconvergence. Boldrin and Montrucchio (1986) established that in such situations "every possible" kind of behavior, including chaotic dynamics, is possible.

Boldrin and Woodford (1990) noted that in the simplest case the discount rates involved imply 10,000 percent real interest rates per period, clearly ridiculous. However, alterations to the model can allow for the phenomenon within more realistic ranges (Neuman, O'Brien, Hoag, and Kim 1988). In particular, Boldrin (1989) has shown that increasing the number of substitutable factors and the ranges of their respective substitutabilities can allow for chaotic dynamics within realistic discount ranges.

Deneckere and Pelikan (1986) have considered further factors beyond the discounting issue that can also bring about chaotic dynamics in one-good, two-sector, optimal infinite horizon, complete markets models. These include the presence of strongly decreasing returns to scale and the possibility that the relative capital intensities of the two sectors reverse order as the level of production rises. This latter condition is key to the arguments of Boldrin (1989) noted above.

Deneckere and Pelikan take the strong position that if optimal behavior is truly chaotic it will never be observed. This is because the learning processes necessary for perfect foresight (and the use of rational expectations more generally) can only operate in relatively regular environments. Sensitive dependence on initial conditions will destroy this possibility and will also render impossible any Grandmont-type optimal governmental policy intervention as well.

Nonseparable Utility and Chaos in Infinite Horizon Models

Yet another case in which chaotic dynamics can arise in optimal infinite horizon models with complete markets is when real money balances enter directly into agents' utility functions in a manner nonseparable from consumption goods. Matsuyama (1991) has examined this case. Let p be prices, c be consumption, and m be real money balances, which grow at a rate μ. Let β be the discount factor and let $\delta = \mu/(\beta - 1) > 0$. Let δ be the elasticity of intertemporal substitution of real balances, with η being a parameter ≥ -1 such that $\delta = (\eta + 2)^{-1}$. Then Matsuyama assumes a utility function of the form

$$U(c,m) = \begin{cases} -[g(c)m]^{-(1+\eta)}/(1 + \eta), & \text{if } \eta \neq -1, \\ \ln g(c) + \ln m, & \text{if } \eta = -1, \end{cases} \tag{11}$$

with $g > 0$ and $g' > 0$.

This implies an optimal time sequence of prices given by

$$(p_{t+1})^{\eta} = (1 + \delta)(p_t)^{\eta}(1 - p_t). \tag{12}$$

Matsuyama proceeds to demonstrate that the nature of these sequences will vary according to combinations of δ and η. An important variable determining the nature of these sequences is

$$\Delta(\eta) = \eta^{-\eta}(1 + \eta)^{1+\eta} - 1. \tag{13}$$

In particular, he shows that for any $\eta > 0$, there exists a value $\Delta^*(\eta)$ satisfying $2\eta < \Delta^*(\eta) < \Delta(\eta)$ such that a period of cycle three and hence chaotic dynamics exists if $\delta > \Delta^*(\eta)$. These results and Matsuyama's more general categorization of possible outcomes are summarized in figure 9.4.

Matsuyama argues that this model suggests the possibility that even for comparative statics analysis the set of possible equilibrium prices may be "topologically complex." This result is probably even more destructive of standard views of economic theory than is the associated undermining of the usefulness of the rational expectations hypothesis.

Empirical Evidence

Much empirical evidence has been gathered that is suggestive of the possible existence of deterministic chaos in various economic time series. However, there has yet to be a definitive demonstration of such existence in any case. Why is this?

Brock and Sayers (1988) have presented a methodology for searching empirically for chaos. The first step is to test for dimensionality and associ-

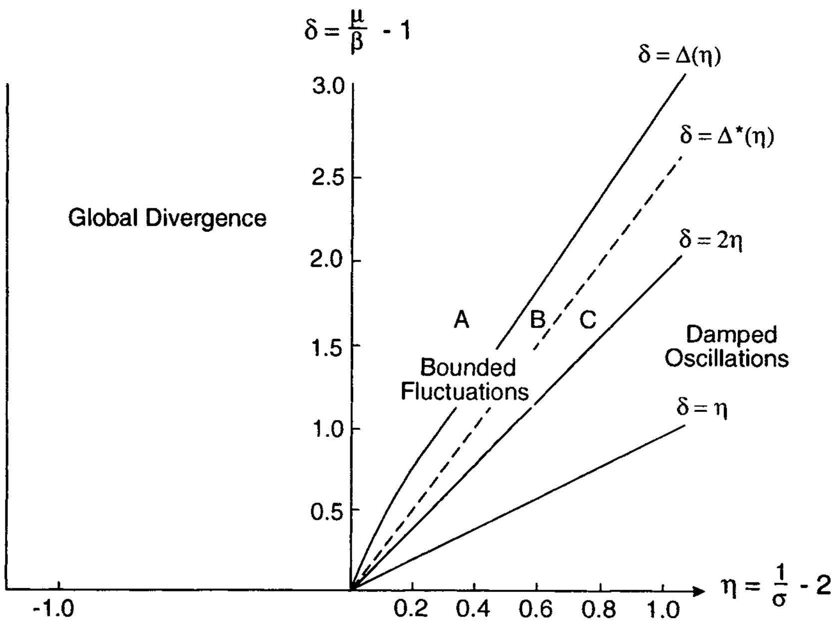

Fig. 9.4. Matsuyama's topological complexity in possible equilibrium prices

ated nonlinearity. Brock, Dechert, and Scheinkman (1987) have developed a test called the BDS statistic that is based upon the use of the Grassberger and Procaccia (1983) correlation dimension. This test has been used on a variety of economic time series and has convincingly shown the presence of serious nonlinearities in many of them. Nonlinear dependence has been found in monetary aggregates (Barnett and Chen 1987), U.S. stock returns (Scheinkman and LeBaron 1989), gold and silver markets (Frank and Stengos 1989), U.S. Treasury bill rates (Brock 1988b), work stoppages (Sayers 1988), employment, unemployment, and pig iron production (Brock and Sayers 1988; Sayers 1990), industrial production (Brock and Sayers 1988; Ashley and Patterson 1989), Japanese GNP (Frank, Gencay, and Stengos 1988), foreign exchange rates (Hsieh 1989; Papell and Sayers 1989), pork production (Chavas and Holt 1991), and milk production (Chavas and Holt 1993).[8]

The second step is to estimate Lyapunov exponents to determine if the greatest exponent has a positive real part, thus indicating the presence of sensitive dependence on initial conditions. Brock and Sayers (1988) argue that although algorithms have been developed for estimating Lyapunov exponents (Wolf et al. 1985; Eckmann et al. 1986; McCaffrey et al. 1992), no statistical inference theory is known, although this is disputed by Barnett and Hinich (1993). Nevertheless, tentative indications of the existence of positive real parts of Lyapunov exponents have been found by Barnett and Chen (1987) for monetary aggregates, by Brock and Sayers (1988) for employment data, and by Eckmann et al. (1988) for stock returns. Why then the continuing empirical skepticism?

The argument has been made that if a series is really driven by deterministic chaos[9] then it should be possible to make short-period forecasts that beat a random walk. Also, the dimensionality should be maintained, even as the intervals of measurement are shortened. Efforts to make such tests on some of the most promising series have not been favorable. Brock and Sayers (1988) could not support the chaos result for any of the series they studied. Efforts to forecast using nearest neighbor techniques failed to beat a random walk for foreign exchange rates (Diebold and Nason 1990) and stock returns (Hsieh 1991). Mayfield and Mizrach (1992) found changing the time interval changed the dimension estimates for stock returns. However, none of these tests can be viewed as definitive and the question remains open.[10]

Indeed, the discussion of the estimation issues involved has been much longer and more complicated than we have indicated here. Let us note one argument that may be of relevance, namely that of aggregation. Almost all these tests have been performed on fairly highly aggregated data sets. However, there is reason to believe that even if chaos exists at the specific or micro-level, it may disappear as aggregation to the macro occurs. Sugihara, Grenfell, and May (1990) found evidence of chaotic dynamics in city-by-city measles epidemics. However, upon aggregating to the national level the dynamics reduced to a noisy two-year cycle. A similar argument has been made with respect to the dynamics of spatial economic systems: that the more local they are the more likely they are to exhibit unstable cyclical dynamics (Dendrinos with Mulally 1985). Thus it may well be that the future search for economic chaos should focus more on highly disaggregated microeconomic series.

The Problem for Rational Expectations

Even if full-blown chaos has not been confirmed for any major macroeconomic time series, many of the problematic aspects of chaotic dynamics still adhere to series that are highly nonlinear while not fully chaotic. We return to our central contradiction. On the one hand, the possible coincidence of chaos and rational expectations could mean the possibility of improved forecasting. On the other hand, the problem of the sensitive dependence on initial conditions means that introducing any stochasticity at all into such a system probably ruins any hope of improving forecasting ability and renders meaningless the very idea of rational expectations.[11] DeCoster and Mitchell (1992) have suggested a possible response to this problem in the form of running many simulations and using the expected value, although this is very time-consuming and expensive.

This problem relates to an old issue that has plagued rational expectations models almost from their inception, namely, the possibility of indeterminacy due to an infinity of possible equilibria (Rosser 1991b). Such a possibility was first shown by Gale (1973) and numerous efforts by New Classical economists have been made since to demonstrate the primacy of the golden rule steady state that the economy is supposed to tend to in such models. Generally the non–golden rule steady state equilibria have the quality of being self-fulfilling prophecies, also known as "sunspot equilibria" (Shell 1977).

A recent line of arguments that has sought to rule out such extra equilibria has been to argue that they are not dynamically stable. If one assumes certain kinds of learning processes then the system will converge on the golden rule steady state. Evans (1985) first presented such arguments and Marcet and Sargent (1989) have further developed them by arguing for the so-called least squares learning process that appears to have this property. But other learning processes lead to different results. Grandmont (1985) has shown learning processes that converge to cycles. Woodford (1990) has shown learning that converges to self-fulfilling prophetic equilibria. Benassy and Blad (1989) argue that learning will not converge on rational expectations unless one starts with rational expectations. Evans and Honkapohja (1992) have found expectationally stable bubble equilibria, and even in his initial study Evans (1985) noted that for certain parameter values of his learning process the result is not convergence but chaotic dynamics.

Is there a way out of this indeterminacy? One such way is to abandon rational expectations in general and to adopt a line of reasoning mentioned above due to Heiner (1989). He argues that when forecasting errors occur, as we would expect in a chaotic environment impinged upon by stochastic shocks, agents will use an adaptive expectations strategy by stabilizing their behavior in the face of heightened uncertainty. He recognizes that for convergence to the optimal trajectory to occur strong assumptions are necessary. But under the right conditions such convergence will proceed slowly at first, then accelerate, and then slow down again as the goal is approached. Rational expectations will only hold in the equilibrium case. Otherwise, simplicity and caution will create order out of chaos.[12] The ultimate irony here is that the deeper concept of rationality may be preserved in such circumstances by abandoning any hope of following rational expectations.

Conclusion

So what is left of the edifice of standard neoclassical theory after the possibility of nonlinear and chaotic dynamics is recognized? As noted above, Mirowski (1990) has argued that in fact chaos theory actually reinforces standard theory. However, this is certainly a minority view. As we have seen, sensitive dependence on initial conditions is profoundly disruptive of the ability to develop rational expectations, especially when any stochastic shocks are present. This is so, or perhaps especially so, when rational expectations equilibria are chaotic, as argued above by Deneckere and Pelikan (1986).

Nevertheless, it probably remains the case that standard neoclassical theory remains useful for many situations despite the difficulties raised by nonlinear and chaotic dynamics. Let us conclude this discussion with a quotation by Smale, one of the initial developers of chaos theory in mathematics, as well as a student of the structural stability of Walrasian general equilibria:

> How did Relativity Theory respect classical mechanics? For one thing Einstein worked from a very deep understanding of the Newtonian theory. Another point to remember is that while Relativity Theory lies in contradiction to Newtonian theory, even after Einstein, classical mechanics remains central to physics. I can well imagine that a revolution in economic theory could take place over the question of dynamics, which would both restructure the foundations of Walras and leave the classical theory playing a central role. (Smale 1977, 95)

NOTES

Figure 9.3 is reprinted from J.-M. Grandmont, "On Endogenous Competitive Business Cycles," *Econometrica* 53 (1985): 995–1045, with permission of The Econometric Society, Department of Economics, Northwestern University, Evanston, IL.

1. The formal definition of rational expectations is that the subjective probability distribution regarding future reality inside the heads of economic actors coincides with the objective probability distribution operating outside their heads in reality.

2. Reviews of applications of chaos theory in economics are in Kelsey 1988; Baumol and Benhabib 1989; Lorenz 1989; Boldrin and Woodford 1990; Rosser 1991a; Brock, Hsieh, and Lebaron 1991; and Bullard and Butler 1993. For formal definitions of chaotic dynamics, see chapters 2, 3, and 4 in this volume.

3. Such a model is potentially applicable to bandwagon phenomena in asset markets such as speculative bubbles. Models of speculative bubbles coinciding with chaotic dynamics have been developed by Day and Hwang (1990); De Grauwe and Vansanten (1990); Ahmed, Ayogu, and Rosser (1990); and De Grauwe, Dewachter, and Embrechts (1993). However, none of these models allow for rational expectations bubbles because of the boundedness conditions associated with chaotic dynamics.

4. Yet another area of microeconomics where chaotic dynamics models have been developed that are inconsistent with rational expectations is in oligopoly theory, especially the case of duopoly dynamics. The first explicit application of chaos theory in economics (Rand 1978) was in such a model.

5. Chaotic macroeconomic models not consistent with rational expectations have been developed depending on nonlinear population dynamics (Stutzer 1980), nonlinear productivity of capital in a Solow-type growth model (Day 1982), high exploitation of labor in the convergence to a Ricardian steady state (Bhadurk and Harris 1987), Lotka-Volterra employment dynamics with markup wages (Pohjola 1981), nonlinear multiplier-accelerator relations (Gabisch 1984), nonlinear investment functions (Dana and Malgrange 1984), Schumpeterian technological long waves (Goodwin 1986), and long-wave socialist investment cycles (Rosser and Rosser 1994), among other approaches.

6. Such outcomes in some economics chaos models have led Mirowski (1990) to criticize them as being sophisticated efforts to preserve neoclassical economics in the face of actual extreme stochastic uncertainty.

7. Woodford (1989) has shown that similar results can be obtained, even in models with infinitely lived agents, if they face short-run borrowing constraints.

8. This nonlinear dependence may take the form of nonlinear dependence in the variance, or conditional heteroskedasticity, rather than in the mean of the series.

9. A very real possibility is a combination of chaos and stochastic shocks. However such a combination will exactly resemble purely random processes because of the sensitive dependence on initial conditions. Every shock will push the system onto a different path.

10. Even though chaos has not been confirmed, nonlinearity most certainly has been. A popular approach to analyzing it in stock and foreign exchange markets has been to examine variations in volatility over time following a suggestion of Mandelbrot (1963). Engle (1982) developed the autoregressive conditional heteroskedasticity

(ARCH) technique to estimate such effects. This has been expanded to generalized ARCH (GARCH) by Bollersley (1986), ARCH in mean (ARCH-M) by Engle, Lillien, and Robins (1987), and nonlinear ARCH (NARCH) by Higgins and Bera (1992).

11. This is not necessarily the case in the long run if there is only one equilibrium. Intermediate run forecasting may be ruined, but the system may still oscillate around some expected value in the long run, which is consistent with the idea of forecasting errors being random.

12. Lavoie (1989) has used neo-Austrian theory to argue that out-of-equilibrium chaotic dynamics will lead to self-organizing order in a manner similar to that described in the nonequilibrium phase transition theory of Prigogine and Stengers (1984).