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CHAPTER 5

From Individuals to Groups: The Aggregation of Votes and Chaotic Dynamics

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> One takes the sum of particular wills . . . [and] . . . take[s] away from these same wills the pluses and minuses that cancel one another . . . [and] . . . the general will remains as the sum of the differences.
>
> —Jean Jacques Rousseau

The aggregation of individual preferences into a group choice is one of the most significant questions in political science. How citizens combine and weigh their interests and desires toward a societal agreement is the foundation of democratic theory. However, the individual-group connection is not as straightforward as early democratic theorists assumed. The process of reaching a societal agreement between a group of individuals, each with their own preferences and abilities to act strategically, is no longer subsumed under a simple additive relation where the group interest is merely the sum of individual interests.

It is now well accepted that the individual-to-group connection is capable of serious pathologies, regardless of the voting scheme used (e.g., Arrow 1963). But establishing the *possibility* of pathologies, such as identifying examples of voting paradoxes, does not imply that the connection between individual and group is fully understood. We know some conditions under which democratic social choice falls short of our ideals, but we do not know why it fails or what set of tools is appropriate to address the "black box" of the aggregation from individual to groups.

This chapter demonstrates that the process of aggregation from individual to group is in the realm of chaotic dynamics. In particular, the nonequilibrium cases of social choice, where group dynamics fail to reach a stable agreement point, exhibit chaos. This observation accounts for the wide diversity of potential group outcomes and the nonadditive aggregation function. These findings also suggest that the tools of chaotic dynamics may shed light on why the instability occurs, where social choice will be most unstable, and how the instability can be reduced. These are future topics. The first task is to establish that the micro-macro connection manifested in social choices exhibits chaos in its nonequilibrium cases. The chapter is organized as fol-

lows. The first section examines the social choice debate and its importance for democratic theory. The next section introduces chaotic dynamics using the example of the logistic function to illustrate the intuition behind examining a chaotic process using symbolic dynamics, in particular the iterated inverse image approach. The third section reviews the multidimensional spatial voting model and the fourth applies the tools illustrated in the second section to the case of multidimensional spatial voting. To suggest the robustness of this connection, the fifth section briefly outlines existing results on additional cases of individual-group aggregations, including simple voting and equilibrium-adjusting market mechanisms. All these related aggregation schemes also have connections to chaotic dynamics.

The sixth section explores an implication of chaotic dynamics in social aggregation functions: the presence of an underlying structure created by the interaction of individuals. Chaotic processes are unusual in that although they exhibit complex, seemingly random possibilities, they are in fact highly constrained. The final section concludes by returning to the issues raised by democratic theory and to the implications of the individual-group connection being chaotic.

From Individuals to Groups

Democratic theory, particularly in its populist form, depends on a fundamental assumption that one can move coherently from individual preferences to a societal choice (Dahl 1956; Riker 1982a). If societal choices have little relation, no relation, or an inverse relation to the desires of its citizens, then the meaningfulness of democracy is called into question. Society becomes an entity that is independent of the individuals that compose it. As Riker (1982a) points out, democracy depends on more than the ends it achieves; it is based on the validity of the means by which it achieves its ends. Voting, as the mechanism by which individual preferences are aggregated into a democratic choice, is at the center of the legitimacy of democratic means.

Early democratic theorists assumed that aggregating individuals' votes was a coherent process. In some cases, the "general interest" was assumed to be equivalent to the citizens' interests. Rousseau assumed that the social contract created a "moral and collective body" that contained the will of the people in what he referred to as the "general will." Similarly, Hegel spoke of the "national spirit" as a coherent single entity. Where addressed specifically, the process of moving from individual to group was viewed as an additive relation: an aggregation or sum. Rousseau outlined a scheme where "pluses and minuses cancel one another." Bentham viewed the connection as a sum of the utilities of individuals.[1] Even as recently as 1958, Truman (1971, 260) spoke of public opinion as "an aggregate of the more or less rational opinions held by the individuals who . . . make up the 'public.'"

Even these early theorists had already conceptualized the individual-group connection in terms of a mathematical function: a mapping rule from a set of individual preferences (or utilities, in the case of Bentham) into a societal choice or a societal preference ordering. Rousseau, Bentham, Truman, and countless others interpreted this "function" in additive terms—the simplest possible mapping. Yet this function need not be additive. As Arrow (1963, 4) points out, there are an infinite number of possible mappings from individual utilities to a social utility function: the sum of individual utilities, or their product, or the product of their logarithms, or the sum of their products taken two at a time, and so on. And even if one assumes that interpersonal comparison of utilities is nonproblematic, there is an implicit value judgment in choosing the aggregation rule.

The assumption that the connection from individual to group is a straightforward additive relation became weakened by the discovery of presumably isolated voting paradoxes. As democracies spread and interest in voting was becoming more common, examples were discovered, such as those by Condorcet, Borda, Dodgson, and Nanson (see Black 1958), where the aggregation of votes led to "peculiar" social outcomes. These outcomes are peculiar in the sense that a different outcome results if the voting order is changed, or the addition or withdrawal of what should be superfluous options causes unexpected and radical changes in the outcomes, such as reversing the social ordering by making the winner the new loser (see, e.g., Black 1958; Farquharson 1969; Fishburn 1973; Ordeshook 1989). The discovery of these examples was not consistent with what was assumed to be an additive relation, for additive relations do not show such variability and sensitivity. However, the implications for democratic theory remained small as long as the examples were viewed as interesting, but isolated, curiosities.

The discovery of cases where the aggregation of individual preferences was problematic led to debates over which voting rule was the "best" in the sense that it would not be subject to a breakdown in the coherence of the mapping from individuals to group. As early as the eighteenth century, Borda and LaPlace were debating and advocating certain voting rules as better in terms of avoiding paradoxes (Saari 1985a). However, Arrow's (1963) General Possibility Theory demonstrated that no procedure exists for passing from a set of individual preferences to a pattern of social outcomes that is consistent with a set of minimally desirable conditions of any democratic process. Although problems with the individual-group connection did not occur for every set of individual preferences (as Black's [1958] single-peakedness condition demonstrates), the possibility of unstable group outcomes did occur with more than a negligible probability (e.g., Niemi and Weisberg 1968; Riker and Ordeshook 1973; Ordeshook 1989).

A further blow to the assumption of a nonproblematic additive relation between individuals and groups came when voting involved two or more

issues (Davis and Hinich 1966). When the topic to be voted on contains—either explicitly or implicitly—two or more components, then stability in a group outcome is nearly impossible (Cohen 1979; Cohen and Matthews 1980; McKelvey 1976, 1979; Schofield 1979, 1980; Plott 1967). In this case, any social outcome can arise from a democratic process, since every outcome can be defeated by some coalition of voters. In addition, voters have a disturbing potential for manipulating the agenda in order to influence societal outcomes (Gibbard 1973; McKelvey 1976).

We now know that the connection between individuals' preferences and group outcomes is not as simple as originally thought. What was once assumed to be a trivial exercise in aggregating or summing individual preferences in an additive way has become a "black box" that has the potential to transform straightforward individual preferences into outcomes that can be very complex and sensitive to small changes. The function that maps individual preferences to group choices is not linear and additive; in fact, as will be seen, the mapping is nonlinear. A nonlinear relationship implies that an increase in one variable does not cause a uniform increase in the other variable: uniform changes in individual preference orderings do not imply uniform changes in the social orderings, as is evident in the winner-turn-loser paradox and the inverted-order paradox. There is a domain for functions of this type: chaotic dynamics. Since the social choice function is nonlinear, it is natural to expect chaos theory—the theory of nonequilibrium, nonlinear dynamics—to apply to the function that aggregates individual preferences into a social choice (see also Schofield 1993).

Chaotic Dynamics

Before examining the dynamics of the mapping from individual preferences to social outcomes using chaos theory, it is worthwhile illustrating the characteristics, methods, and formal definition of chaos with a simple example from outside of choice theory. This section introduces the concepts of chaos theory by exploring the well-known logistic function. The example is important because it illustrates the definitions and techniques used later in the chapter. The approach uses symbolic dynamics (Devaney 1989) and, in particular, the iterated inverse images (Saari 1991). Using this approach makes demonstrating Devaney's (1989) three requirements of chaos, sensitive dependence on initial conditions, topological transitivity, and dense periodic points, relatively straightforward. In a subsequent section, these techniques will be applied to the social choice function of multidimensional spatial voting.

One of the simplest functions that exhibits chaos is the logistic equation, discussed in chapter 2. The logistic function illustrates the three important characteristics of a function exhibiting chaos: (1) sensitive dependence on initial conditions, (2) topological transitivity, and (3) dense periodic points.

Devaney (1989) defines the presence of chaos in a function by the satisfaction of these three conditions:

DEFINITION 1. *A function exhibits sensitive dependence on initial conditions if two points that are arbitrarily close separate by a distance* δ *by the iteration of that function (Devaney 1989, 49).*

DEFINITION 2. *A function exhibits topological transitivity if the mapping has points that move from one arbitrarily small neighborhood to another arbitrarily small neighborhood (Devaney 1989, 49).*

DEFINITION 3. *A function has dense periodic points if, given a periodic point, there is another periodic point arbitrarily close by (Devaney 1989, 15, 42).*

These conditions combine for the formal definition of chaos (Devaney 1989, 50):

DEFINITION 4. *A function is chaotic if it satisfies sensitive dependence on initial conditions, topological transitivity, and dense periodic points.*

These three conditions have all been informally illustrated with the example of the logistic function in chapter 2. *Sensitive dependence on initial conditions* is evident in the effect of a new initial condition. *Topological transitivity* implies that one can move from one region of x to another region—i.e., that outcomes do not merely repeat the same few values but have a diversity of potential outcomes. *Dense periodic points* are evident in the bifurcation diagram. Periodic points are nearby other periodic points; there is a consistency in nearby trajectories. Although all three conditions are intuitively evident in the example of the logistic function, the proof that these conditions are in fact satisfied requires more. Establishing these three conditions formally becomes straightforward when the tools of symbolic dynamics are used.

Symbolic Dynamics

The intuition behind the presence of cycles and chaos in a function becomes clearer with the tools of symbolic dynamics (see, e.g., Devaney 1989; Saari 1991). This technique permits a cataloging of the permissible sequences of the function. It accomplishes this by abandoning point precision, as the previous discussion emphasized, and examining instead how regions map into regions. In particular, this approach reverses the focus to look at how sets of points map to other sets, rather than at where a single point maps.

To illustrate this approach, divide $x_t = [0, 1]$ into two regions, labeled R_1

and R_2, as shown in figure 5.1. The image of these regions is shown on the x_{t+1} axis by reflecting across the $x_t = x_{t+1}$ line. One is interested in which regions map to which regions via the function. In this case, the function is the one-to-one mapping of the logistic function.

Note that from figure 5.1 it is apparent that region 1 maps to region 2 because for some values of $x_t \in R_1$, the function results in some values of $x_{t+1} \in R_2$. Similarly, some values of $x_t \in R_2$ map to region 2. By a similar logic we can describe the rules of mapping under the logistic function for all the regions, summarized as

i) if $x_t \in R_1$ then $x_{t+1} \in R_2$;
ii) if $x_t \in R_2$ then $x_{t+1} \in R_1$ or R_2.

These "rules" on the sequences of mappings outline some of the permissible sequences. Each of these sequences is called a word, and the set of all permissible words under a particular function is called the dictionary. Each sequence describes the iteration of a single starting point via that function. Sequences that are identical for the first n entries and that differ after the $n + 1^{\text{th}}$ entry designate initial points that are nearby. In other words, the distance between two points is measured by the extent to which their sequences match (see, e.g., Saari 1991). The possibility of cyclic sequences arises because of the feedback region of R_2. By iterating within R_2, cycles of all lengths are possible; the mapping back to R_1 allows for the completion of the cycle. The rule on the mapping is simply that R_1 must be followed by R_2; R_2 can be followed by either R_1 or R_2.

For example, a four-cycle sequence is possible: (R_1, R_2, R_2, R_2, R_1, R_2, . . .). Finding this four-cycle sequence is not obvious with the forward approach used in choice theory. The iterated inverse image approach (Saari 1991) identifies where the cycle must appear. First, find all points in R_1 that map to the first target region of $T_1 = R_2$; this is given by $f^{-1}(T_1) = f^{-1}(R_2) \cap R_1$. However, the goal is not to map to any point in R_2, but to the subset of points in R_2 where the next iterate stays in R_2. This means we need to refine the starting region of R_1 to the refined target of $T_2 = f^{-1}(R_2) \cap R_2 \subset T_1$. These points $f^{-1}(T_2) \subset f^{-1}(T_1)$ are all the points that start in R_1, go to R_2, and on the second iterate remain in R_2.

The same argument ensures that the third iterate is in R_2. Here the refined target region is $T_3 = f^{-1}(f^{-1}(R_2) \cap R_2) \subset T_2$, and $f^{-1}(T_3) \subset f^{-1}(T_2) \subset f^{-1}(T_1) \subset R_1$. At each stage, the set of initial points is refined to include only those points that hit the refined target—this is the region that ensures that more of the steps of the proposed cycle occur. The final stage, $T_4 = f^{-1}(\ldots (f^{-1}(R_1)) \subset T_3$, defines the set of inverse iterated points $f^{-1}(T_4) \subset R_1$. This is the set of all points in R_1 where the image of the fourth iterate is in

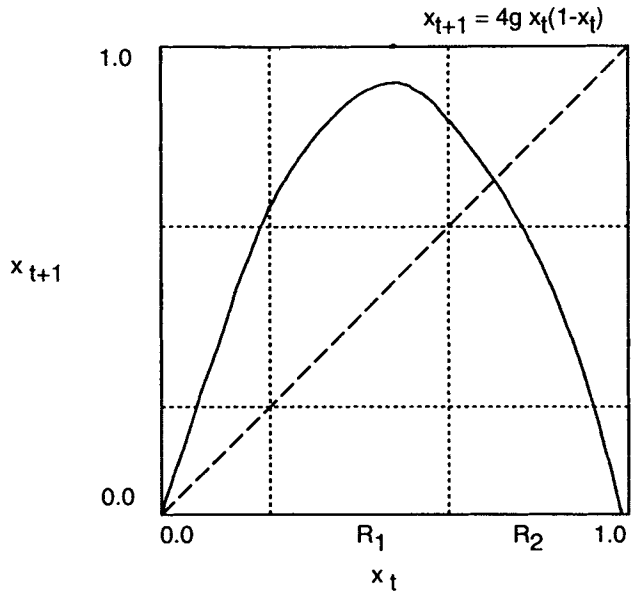

Fig. 5.1. Map of the logistic function

R_1. Trivially, there is at least one point $x \in R_1$ so that $f^4(x) = x$. This is the period-four orbit.

The possibility of cyclic points and of a set of nonperiodic or chaotic points arises because of the presence of a feedback in the logistic function. It is the fact that region one maps to region two, which maps back to region one, that allows for the abundance of permutations of sequences. A similar feedback process occurs in nonequilibrium social choice, evident in the presence of an intransitivity among three alternatives.

Using symbolic dynamics, Devaney's three conditions for chaos can easily be shown to be satisfied, verifying that the logistic function is indeed chaotic. To establish sensitive dependence on initial conditions, one must demonstrate that two points that are arbitrarily close will separate by a distance δ by the iteration of the function. This requires that one can construct two sequences of symbols 1, 2 that are identical for n entries and that eventually differ in their $n + 1^{\text{th}}$ entry. Let the distance between regions 1 and 2 be δ. Assume that the nth entry of these two sequences is 1. Then the $n + 1^{\text{th}}$ entry can be 1 for the first sequence and 2 for the other, separating the points by δ and demonstrating sensitive dependence on initial conditions.

Topological transitivity requires that the mapping have points that move from one arbitrarily small neighborhood to another arbitrarily small neighbor-

hood. Then any point that begins in one and ends in two satisfies this condition. It is straightforward to come up with a permissible sequence based on the mapping rules of the logistic such that the two sequences end in different symbols.

The condition of dense periodic points requires that, given a periodic point, there is another periodic point arbitrarily close by. Recall that distance between the iteration of points is evident by the extent to which two sequences match. Given a sequence of arbitrary length, it is possible to find another sequence that matches the beginning n entries of the first sequence, demonstrating the existence of two periodic points arbitrarily nearby. This is obvious in terms of the permissible sequences of 1, 2. For example, the initial conditions leading to (1222222) and (1222221) would be nearby points.

The satisfaction of these three conditions verifies that the logistic function is chaotic. Devaney (1989, 50) summarizes these conditions in terms of three characteristics that a chaotic function possesses: unpredictability, indecomposability, and an element of regularity. The presence of sensitive dependence on initial conditions makes long-term prediction of a chaotic process impossible. Chaotic systems are indecomposable because they cannot be broken down into two subsystems that do not interact; this arises because of topological transitivity. Third, although a chaotic process exhibits complex, apparently random, behavior, there is an element of regularity, in that periodic points are dense and nearby points behave similarly.

Multidimensional Spatial Voting

These same techniques can be applied to the case of voting by considering the social choice function as a mapping rule. This section summarizes some of the notation and main results of multidimensional spatial voting (see Krehbiel 1988 or Strom 1990 for a complete survey), and can be skipped by those familiar with the standard notation and approach.

The choice among alternatives is often modeled spatially (Davis and Hinich 1966), where the alternatives to be voted on are arranged on a continuum, such as the liberal-conservative spectrum or the amount of a budget allocation. The one-dimensional spatial approach has led to several interesting implications, notably Black's (1958) median voter theorem, stating that if all voters' preferences are single-peaked then a stable outcome emerges at the median voter, and Downs's (1957) results on party competition and the tendency for parties to move toward the center of the distribution of voters. The single-dimension continuum can be extended to include voting over several policy dimensions, such as when an issue implicitly contains two or more components or when issues are linked and must be decided as a package.

Assume an alternative space of issues with m issue dimensions. Assume

a set $N = \{1, 2, \ldots n\}$ of voters where each voter has preferences over the issues in the policy space. Each voter's most-preferred outcome is referred to as his or her ideal point. Each voter also has preferences over the entire issue space that are typically assumed to be a monotone decreasing function of the distance between a proposed point and the voter's ideal point. Then each voter's indifference curves are spheres around his or her ideal point. These utility functions are referred to as Type I utility functions and can be summarized as

$$x \succeq_i y \Leftrightarrow \| x - x_i \| \leq \| y - x_i \|, \tag{1}$$

or, alternative x is preferred to (or indifferent to) alternative y by voter i if and only if the distance between x and i's ideal point x_i is less than (or equal to) the distance between proposal y and i's ideal point. Simply put, the rule on preference among alternatives compared to a voter's ideal point is simply "the closer the better."

For simplicity, I restrict attention to simple majority rule. A set of voters, M, is a majority-rule winning coalition if

$$|M| \geq \frac{n+1}{2} \quad \text{if } n \text{ is odd and } |M| \geq \frac{n+2}{2} \quad \text{if } n \text{ is even.}$$

Let f denote the social choice function for simple majority rule. Then $f(x) = \{y \mid y \succ x\}$ is the set of points in the policy space that are majority-preferred to x. In other words, $f(x)$ is the set of points that beats x by some majority coalition. If $f(x) = \varnothing$, then there are no proposals that beat alternative x and x is the winner. Any $x \in X$ for which $f(x) = \varnothing$ is called a core point. Note that if a core point exists then the iteration of the social choice function will reach an equilibrium at outcome x.

However, the existence of an equilibrium under the social choice function of majority rule in a multidimensional policy space is extremely rare. In order to achieve a majority rule equilibrium outcome, severe restrictions on the voters' ideal points are required (Black 1958; Plott 1967). For nearly all voter preferences, majority rule does not lead to any stable winning social outcome, but continues to wander over the policy space. Every issue can be beaten by some other proposal that is also preferred by a majority of the voters. In addition, the movement of winning proposals can exhibit intransitivity or cycles, where, by a sequence of majority votes, the social outcomes will be alternative b over a, c over b, and then a over c! Moreover, any point in the policy space can be reached by some sequence of majority rule decisions. When an equilibrium breaks down, it breaks down completely and includes the entire policy space (Cohen 1979; Cohen and Matthews 1980; McKelvey 1976, 1979; Schofield 1979, 1983). These results are typically

referred to as the *chaos theorems*, because of the negative interpretation that all order and stability are lost: agendas can be constructed between any two points, sequences of votes do not reach a stable equilibrium consensus, and any final group choice may have little relationship to the voters' preferences. However, the use of the term *chaos* was semantic. The following section demonstrates that the connection between multidimensional spatial voting and chaos is not merely semantic, but theoretic.

Chaotic Dynamics in Multidimensional Spatial Voting

The previous section briefly summarized some of the troubling aspects of multidimensional voting and pointed out the inherent and pervasive instability in voting. By viewing the social choice process as a nonlinear mapping, symbolic dynamics can be used to establish that the sequences generated by a group of voters in a multidimensional policy space do in fact exhibit chaotic dynamics (Richards 1992, 1994). The chaos theorems of social choice can be understood as chaotic nonlinear dynamics; the connection is not merely semantic, but is theoretic. This implies that outcomes and agenda paths are not completely random and all order is not lost.

This section demonstrates the existence of chaotic dynamics in multidimensional spatial voting using an approach analogous to the example of the logistic function shown in the second section. However, there is an important technical distinction. In the case of the logistic function, the mapping was one-to-one and continuous. In the case of multidimensional social choice, the "function" is instead a *correspondence*: each policy point maps to a *set* of points that beat it by majority rule. However, the general logic remains the same: divide the space into regions and examine how regions map into other regions. The rules of the mappings catalog the feasible dynamics and allow for the establishment of Devaney's three conditions of chaos.

The Importance of Cycles

Before examining the inverse image of the social choice function,[2] two preliminary facts must be established. First, every multidimensional spatial voting context without a core has at least one three-cycle among alternatives. Second, if a cycle of any length greater than three exists, then a three-cycle also exists. These results are similar to previous findings (e.g., Cohen 1979; Schofield 1983), but are presented using concepts needed for the subsequent results. First, the notion of a cycle must be clarified. An alternative $y \in X$ can be reached from a point $x \in X$ if it is possible to move from x to y by iterations of f. If an $x \in X$ can be reached from itself by k iterations of f and $x \neq f^j(x)$ for $1 \leq j < k$, then a k-cycle exists.

(1) For two or more issue dimensions, three or more voters, type I utility functions, and majority rule, then if the core is empty, there exists a sequence of alternatives (a, b, c) such that $b \in f(a)$, $c \in f(b)$, and $a \in f(c)$.

To see that this is true, suppose that a three-cycle exists among three alternatives, a, b, and c. Then a, b, and c form a nondegenerate triangle in R^m. Label the three perpendicular bisectors of $\overline{ab}$, $\overline{bc}$, and $\overline{ca}$ as H_{ab}, H_{bc}, and H_{ca}, which must intersect in a space of $m - 2$. Denote H^+_{ab} as the open halfspace that includes b and that by construction contains a majority of voter ideal points. Denote H^+_{bc} and H^+_{ca} similarly, such that each halfspace contains c and a, respectively, and contains a majority of voters. A cycle exists among three points a, b, and c if such a partitioning is possible (fig. 5.2). This partitioning creates the "pinwheel" of social choice preferences such that $b \in f(a)$, $c \in f(b)$, and $a \in f(c)$. Therefore, the question of the existence of a three-cycle for a given configuration of voters is the question of whether voters' ideal points can be partitioned into the appropriate pinwheel of hyperplanes.

Assume voters' ideal points are located in the policy space such that a core exists. Denote the core point as v^*. Then $v^* \in f(x)$ for all $x \neq v^*$ and $f(v^*) = \varnothing$. The agreement set of any winning coalition of voters must contain v^* or there would be a winning coalition that could beat v^*, violating the construction of v^* as a core point. Therefore, in partitioning the voters into the sets H^+_{ab}, H^+_{bc}, and H^+_{ca}, v^* must be an element of H^+_{ab}, and of H^+_{bc}, and of H^+_{ca},

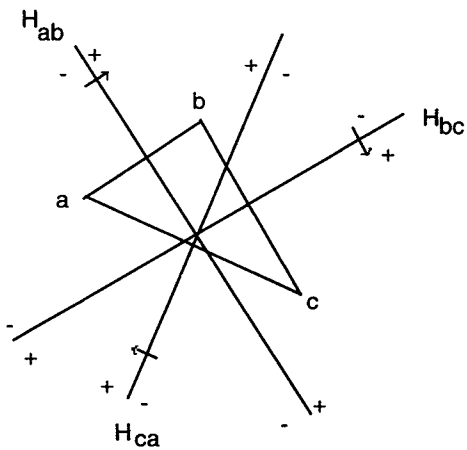

Fig. 5.2. Three-cycle pinwheel partition

implying that $H^+_{ab} \cap H^+_{bc} \cap H^+_{ca} \neq \varnothing$. However, for a three-cycle pinwheel partition to exist, the hyperplanes must be such that $H^+_{ab} \cap H^+_{bc} \neq \varnothing$ and $H^+_{ab} \cap H^+_{bc} \cap H^+_{ca} = \varnothing$ (see fig. 5.2). Therefore, if a core exists, a three-cycle partition is not possible.

Conversely, assume voters' ideal points are located in the policy space such that a core does not exist. If the core is empty, then the halfplanes H^+_{ab}, H^+_{bc}, and H^+_{ca} must be able to be placed such that a majority of voters are contained in each H^+ and such that $H^+_{ab} \cap H^+_{bc} \cap H^+_{ca} = \varnothing$. If this were not the case, then there would be a point that must be included in every winning coalition's set and this point would be a core. Given that there is some H_{ca} such that $H^+_{ab} \cap H^+_{bc} \cap H^+_{ca} = \varnothing$, then there must be an H_{ca} such that the three lines intersect in a single point. Either the three lines already intersect in a single point or H^-_{ca} includes all of the cone $H^+_{ab} \cap H^+_{bc}$. In the second case, H_{ca} has the flexibility to move toward the vertex of the cone (toward H_{ca}'s "$-$" region), since this can only result in more, rather than fewer, voter ideal points. Therefore, H_{ca} can be moved until it intersects H_{ab} and H_{bc} in a single point at the base of the $H^+_{ab} \cap H^+_{bc}$ cone. Therefore, if there is no core then a three-cycle partition is possible.

> (2) If there is a k-cycle for a $k \geq 3$, then there is some a', b', and c' such that $b' = f(a')$, $c' = f(b')$, and $a' = f(c')$, i.e., there is a three-cycle.

To see that this is true, note that the existence of a k-cycle implies that the core is empty (Cohen 1979; Schofield 1983). By (1), if the core is empty then a three-cycle exists.

The Iterated Inverse Image of the Social Choice Function

The previous discussion established that when voters' ideal points are such that no equilibrium exists, then the majority-rule social choice function creates a mapping from a to b, from b to c, and from c to a. As in the example of the logistic function, the existence of regions that feed back into previous regions allows for the existence of cycles of all lengths and for the demonstration of chaos. Therefore, one can characterize the permissible sequences of the social choice function in terms of mappings of regions. This is accomplished by examining the iterated inverse image of the social choice function. The policy space is partitioned into regions, and by examining the nested sequences in terms of target regions, one can outline the permissible sequences under the iteration of the social choice mapping.

This approach differs from the typical approach to spatial voting in two

ways. First, rather than examining the mapping of single points—as in the "win set" formed by the indifference curves through a single point—this approach examines the mapping of regions into regions. By abandoning the point precision of previous approaches, one can characterize the general dynamics of the social choice function (Saari 1991). Second, the iterated inverse approach differs from previous analyses of spatial voting by focusing not on the social choice function, but on its inverse, $f^{-1}(x)$. Previous approaches examined the set of points that beat an existing status quo point, namely, the set $f(x)$. The iterated inverse image approach reverses this logic and examines the target region of x—namely, the set of points that map to x via the social choice rule. If y is the target region of x, then x can follow y in a social choice sequence. The iterated inverse image approach outlines which sequences of outcomes are possible under f by examining the iterations of target regions. This allows for a general cataloging of everything that can happen under the social choice rule in terms of permissible sequences of target regions.

PROPOSITION 1. *For two or more issue dimensions, three or more voters, type I utility functions, and majority rule, if a three-cycle among alternatives exists, then there are cycles among alternatives of all periods greater than or equal to three.*

PROOF. Let a, b, and c denote three alternatives in a three-cycle, with H_{ab}, H_{bc}, and H_{ca} the respective perpendicular bisectors (as in fig. 5.2). Label a closed region B bounded by a hyperplane parallel to H_{ab} and such that $b \in B$. Define a set $R(B)$ as the open set of all points defined by reflecting across H_{ab} (see fig. 5.3). By construction $a \in R(B)$, since H_{ab} is the perpendicular bisector and $b \in B$. Regions C, $R(C)$, A, and $R(A)$ are defined similarly, such that $c \in C$, $b \in R(C)$, $a \in A$, and $c \in R(A)$ (see fig. 5.3).

Assume a sequence $\{A, B, B, \ldots, A, \ldots\}$. First we need to find those points in A that map to $T_1 = B$ in the first iterate. This is given by the cone $f^{-1}(T_1) = R(B) \cap A$. Now we need to refine $f^{-1}(T_1)$ to only those points that remain in B in the second iterate. That is, we want to find the points in A that map to the refined target region of $T_2 = f^{-1}(B) \cap B \subset T_1$. The open cone $B \cap H_{bc}^{-} \in f^{-1}(B)$, since all points in this cone map to B by approaching the H_{bc} hyperplane, so $T_2 = B \cap H_{bc}^{-}$. The same argument ensures that the third iterate is in B. Here the refined target region is $T_3 = f^{-1}(f^{-1}(B) \cap B) \subset T_2$. At each iteration, the set of initial points is refined to include only those points that remain in B. Then $T_{i-1} = f^{-1} \ldots (f^{-1}(B) \cap B)$; one can stay in B as many times as necessary since the boundary H_{bc} is open. The final stage defines the set of inverse iterated points $f^{-1}(T_i) \subset A$—i.e., the set of points in

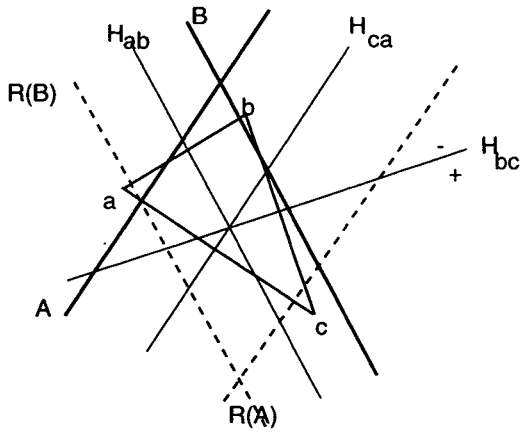

Fig. 5.3. Three-cycles within three alternative choices

A where the image of the ith iterate returns to A. This means the final target region must be refined to $T_i = H_{bc}^{-} \cap f^{-1}(A)$. This is an open cone and is nonempty. Therefore, one can construct a sequence that begins in A, iterates within B for any number of iterations, and then returns to A.

Since sequences of the type $\{A, B, B, \ldots, A, \ldots\}$ are permissible under this social choice rule, it is straightforward to see that cycles of all length greater than three can be constructed: vary the number of iterates with B from two (a three-cycle) to $B = i$ (an $i + 1$-cycle). Note, however, that unlike the logistic function, the presence of a three-cycle feedback does not imply cycles of all lengths in the social choice function.[3] "One-cycles" (i.e., equilibrium points) are not implied. Although region B can follow region B, in fact, subsequent points in B cannot return to previous points. As seen in the above discussion, mappings within B must approach the open boundary of H_{bc}. Therefore, there is no periodic point of one.

Proposition 1 makes clear that the mapping rules for the social choice function allow region B to follow A, C to follow B, and A to follow C. In addition, note that each region can follow itself, merely by moving closer to its perpendicular hyperplane. Therefore, the mapping rules on the majority-rule social choice function can be summarized for x_t as the status quo alternative

$$\begin{aligned} &\text{i) if } x_t \in A \text{ then } x_{t+1} \in A \text{ or } B \text{ or } C;\\ &\text{ii) if } x_t \in B \text{ then } x_{t+1} \in A \text{ or } B \text{ or } C;\\ &\text{iii) if } x_t \in C \text{ then } x_{t+1} \in A \text{ or } B \text{ or } C. \end{aligned} \tag{2}$$

Chaotic Dynamics

Recall from the example of the logistic function that the presence of chaotic dynamics is established by demonstrating satisfaction of three conditions: sensitive dependence on initial conditions, topological transitivity, and dense periodic points. As in the case of the logistic function, demonstrating that the social choice function is chaotic becomes a straightforward exercise when the permissible sequences in terms of symbolic dynamics are used.

PROPOSITION 2. *For two or more issue dimensions, three or more voters, type I utility functions, and majority rule, if an intransitive social choice cycle exists among any alternatives, then the social choice function exhibits chaotic dynamics.*

Recall that to demonstrate sensitive dependence on initial conditions, one must demonstrate that two points that are arbitrarily close will separate by a distance δ by the iteration of the function. Let the distance between regions A and B be δ. Then sensitive dependence on initial conditions implies that one can construct two symbolic sequences that are identical for n entries and that differ on the $n + 1^{\text{th}}$ entry. Assume that the nth entry of these two sequences is A, then the $n + 1^{\text{th}}$ entry can be A for one sequence and B for the other, separating the points by δ and demonstrating sensitive dependence on initial conditions.

Recall that topological transitivity requires that the mapping have points that move from one arbitrarily small neighborhood to another arbitrarily small neighborhood. Then any point that begins in A and ends in B satisfies this condition. Clearly such a point does exist, by the permissible sequences outlined in (2).

Recall that the condition of dense periodic points requires that, given a periodic point, there is another periodic point arbitrarily close by. Distance between the iteration of points is evident by the extent to which two sequences match. Given a sequence of arbitrary length, it is possible to find another sequence that matches the beginning n entries of the first sequence, demonstrating the existence of two periodic points arbitrarily nearby.

By examining the social choice correspondence of multidimensional spatial voting in terms of the iterated inverse images, the demonstration of the existence of chaotic dynamics becomes straightforward. The existence of a three-cycle intransitivity arising from a nonequilibrium preference configuration creates a feedback process among alternatives. It is this nonlinear feedback that, as in the case of the logistic function, creates complex behavior in the mappings, allowing for countless cyclic outcomes and nonperiodic, chaotic sequences.

Chaotic Dynamics in Other Individual-Group Aggregations

Multidimensional spatial voting is one subset of the larger class of aggregation schemes from individuals to groups. The multidimensional case is particularly interesting however, since virtually all voter preferences lead to chaotic dynamics. This section addresses the question of the general robustness of chaotic dynamics in the aggregation from individual tastes to social outcomes. To do this, I briefly review two other aggregation schemes where the existence of chaotic dynamics has been established. These include Saari's (1984, 1985a, 1987, 1989) work on simple voting and the findings of chaotic dynamics in market mechanisms.

Simple Voting

Unlike multidimensional spatial voting, in simple voting individuals rank a finite set of discrete one-dimensional alternatives. Each voter has a preference ordering over the alternatives and the social choice is determined using a given voting rule, such as pairwise majority votes, two-thirds rule, or the Borda count.

As in multidimensional voting, one would like the function mapping individual preferences to a group choice to be coherent in some sense. Arrow (1963) formalized the notion of coherence in terms of five conditions: (1) every individual preference ordering that is complete and transitive is admissible,[4] (2) if all voters prefer an outcome x over an outcome y then the social ordering must be x over y, (3) if x and y are the only two outcomes that the group can consider, then the social preference between x and y depends only on individual preferences over $\{x, y\}$, and not on individual preferences over a larger set, (4) no single voter is decisive for every pair of outcomes, and (5) the additional requirement that the social choice ordering must also be complete and transitive. Arrow showed that no voting rule exists that satisfies these conditions. No matter what voting rule is used, the mapping from individual to group will not be trouble-free. Paradoxes in individual-to-group mapping will occur for every voting rule.

The mapping from individual to group in the case of simple voting also exhibits chaos, as demonstrated by Saari (1984, 1985a, 1987, 1989, 1990, 1992). Using symbolic dynamics, and for any number of alternatives and any positional voting procedure, Saari catalogs all possible social choice rankings over all possible subsets of candidates. He shows that any social choice ordering—no matter how paradoxical (such as the social ordering $a > b$, $c > a$, $a > b = c$)—can be achieved by some profile of voters. Any outcome is

possible, paradoxes are abundant, and it is impossible to construct a voting rule that avoids these paradoxes.[5] This occurs because all sequences are possible under the individual-group mapping rules. The permissible mappings in any positional voting procedure create the existence of chaotic dynamics in simple voting contexts.

Market Mechanisms

In addition to voting schemes, individual preferences can also be aggregated using market mechanisms. As Arrow (1963,1–2) and Riker and Ordeshook (1973) point out, voting and the market are special cases of the general category of collective social choice. Both market mechanisms and voting schemes take the preferences and desires of many individuals and "amalgamate" them into a single social choice. In the case of voting, the choice is a selection of a candidate or policy. In the case of the market, individual preferences over quantity and prices of goods translate into a single market price. In this way, the market incorporates individual preferences and results in a social choice in terms of the allocation of goods among the individuals (Riker and Ordeshook 1973). However, in their acts of voting, individuals are consciously choosing and anticipating a social outcome. In the act of a market mechanism, the act is less consciously oriented toward the ultimate social outcome (Riker and Ordeshook 1973, 82). Nonetheless, the collective outcome of a market mechanism, such as the price of a commodity, is still achieved by an aggregation rule that takes individual preferences over the commodities and leads to a social outcome by the combination of individual actions.

As in voting, there are various rules by which market mechanisms can respond to individual preferences. Here I focus only on the most general form. The basic idea is that one has an exchange economy with a fixed number of commodities, a fixed amount of goods, and a set of agents with given initial endowments. Each individual has preferences over each good, where an individual's demand is given by his or her utility function. The combined demand of all the individuals is the excess demand function. Assume that *total buy* is not equal to *total sell*. The market mechanism is a force that will move prices to an equilibrium point where total buy equals total sell and the market clears. Hypothetically, this is modeled as being adjusted by a *Walrasian auctioneer*, whose mission is to search for the market clearing price (see, e.g., Varian 1992). Prices move iteratively in the direction of excess demand: if the ith commodity is in demand, then its price increases until demand equals supply and it clears. In general, these models are described as *tatonnement* models or price adjustment mechanisms. Depending on the de-

mand functions, the individuals' preferences will combine through the excess demand function and the market rule to yield an equilibrium price and thereby allocate goods among individuals.

However, it is well known that such market mechanisms need not converge to a stable social outcome. For example, economists have long known that price adjustment mechanisms are not stable for two or more commodities. The presence of chaotic dynamics in iterative market adjustments has also now been established (see, e.g., Day and Pianigiani 1991). Saari and Simon (1978) and Saari (1985b, 1992) show that for most prices there need not be any relationship among the demand functions. The outcome of price models can lead to many outcomes in terms of price and allocation, and the social outcome is very sensitive to changes in the set of agents, the set of commodities, or the set of preferences. In addition, no market procedure satisfies three minimal conditions in moving from aggregate excess demand to a market price (Saari 1991). Majumdar (1992) and Bala and Majumdar (1994) show that with two commodities, two agents, and the price of one good fixed, chaotic dynamics will result, as it will in a competitive economy with two commodities.

Constraints on Instability: The Order in Chaotic Dynamics

Interpretations of a chaotic process in terms of the colloquial use of the word *chaos*—as disorderly, random, or conflictual—ignore the unique aspects of chaotic dynamics. A chaotic process is disorderly: it results in an infinite array of seemingly random possibilities; it is sensitive to very small shocks; it never reaches a stable equilibrium; and it is unpredictable in the long term, no matter how long its behavior is empirically observed. However, the other side of chaos is the presence of an underlying order in the dynamics. Although the outcomes of a chaotic process are so complex they appear random, in fact, chaotic processes incorporate specific constraints on these outcomes. It is this mix of complexity and order that makes chaos theory an unusual theoretical framework. In this section we explore some preliminary evidence of this structure in the case of multidimensional spatial voting and discuss its implications for understanding the individual-group aggregation process.

The Constraints on Outcomes and Paths

It is well known that in a multidimensional setting, any outcome in the policy space—even points that are very far away from the voters' ideal points—can be reached by a sequence of majority-rule votes. However, this result is often

interpreted to imply more than it technically does. While it is true that any point can be reached from any other point, this does not imply that any point can be reached from any other point by any path. In effect, for any two origin and destination points, "you can get there from here," but the routes from here to there may be quite constrained.

The symbolic dynamics approach emphasizes the constraints on feasible sequences. The mapping rules outline what is permissible and what isn't permissible in the social choice dynamic. If majority rule was in fact total disorder, then every region would map to every other region *even as regions were refined to smaller and smaller sets.* Although the construction of the fourth section did have the property of the dictionary being the universal set, it also contained only three partitions. It is obvious that as the regions are refined, rules on the mapping will emerge and the dictionary will no longer be the universal set. These constraints were already evident in terms of mappings within a region. Although, for example, B maps to B, in fact, points must approach the hyperplane with each iteration: not all points in B map to all other points in B in one iteration. A refinement of the region B would highlight these mapping rules.

Another example of the order in a chaotic process is evident in the condition of dense periodic points. Recall that this condition states that for any periodic point there is another periodic point arbitrarily close by. In the case of the logistic function, the order in terms of nearby dynamical behavior was evident in the bifurcation diagram, where two-cycles transformed to four-cycles, then a chaotic region, and so on. Bifurcation diagrams can also be constructed of the multidimensional spatial voting social choice dynamics. Like the bifurcation diagram of the logistic function, the bifurcation diagrams of the social choice function show complex patterns (fig. 5.4).

The bifurcation diagrams of figure 5.4 show the outcomes in one of the issue dimensions (y on the vertical axis) as a function of the winning coalition choosing either an *extreme* point (near the tip of the coalition agreement set) or an *incremental* point (near the status quo point—i.e., the base of the coalition agreement set). For many steps over this parameter continuum, a sequence of eighty points was generated and the last fifty points were plotted in a horizontal line. For example, part b of figure 5.4 shows a three-cycle for some of its region. As seen by comparing parts a and b of figure 5.4, a different fine structure emerges as a result of the voting rule and the individual preferences. Each set of voters produces a unique bifurcation diagram, constant to transformations that preserve the configuration of the voters' ideal points (such as rotating or a linear transformation). Although these diagrams are of limited use (because of the nature of the parameter choice to make the function one-to-one), they do illustrate the fine structure to the dynamics of the social choice function.

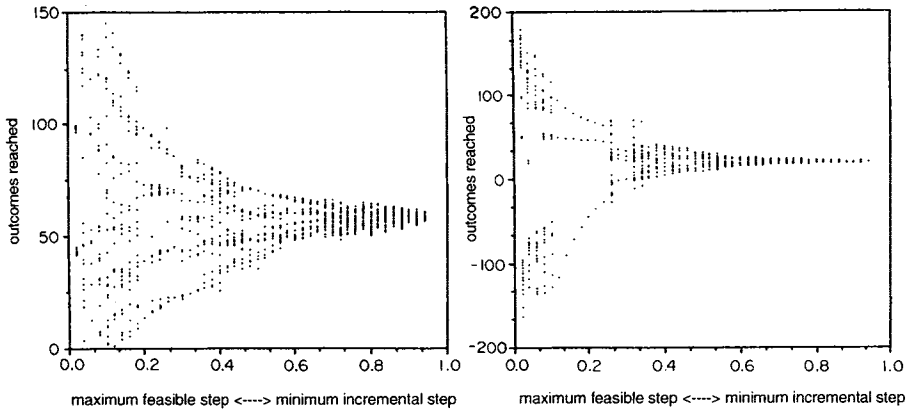

Fig. 5.4. Two sample bifurcation diagrams. Distribution of voter ideal points is: (10, 10) (20, 10) (35, 20) (20, 80) (70, 30) and (60, 30) (60, 70) (70, 50) (90, 70) (120, 110).

The Structure of Outcomes

Given that the social choice function for multidimensional spatial voting is a chaotic process, we know from the definitions of chaos that there must be a strict structure to the feasible outcomes. There are suggestions of this structure in a simulation of iterations of majority-rule decisions.

Figure 5.5 shows the distribution of outcomes with five voters deciding by majority rule over a two-dimensional issue space. Since majority rule is not one-to-one, for the simulation a single point had to be chosen as the winner. It was arbitrarily set as the midpoint of the largest winning coalition's agreement set. Note that the choice of this point is arbitrary—it merely serves as an anchor on the movement of winning coalitions. The simulation was begun with an arbitrary initial point, a_0. a_0 is beaten by a_1, which is the alternative that is preferred to a_0 by the largest winning coalition, where a_1 is the midpoint of that coalition's agreement set. Now a_1 is also not stable and can be beaten by a new set of points by a reformation of coalitions. a_2 is the point that beats a_1 and is the midpoint of the largest coalition that can outvote a_1. Continuing this iterative process provides a sample of the distribution of alternatives under the majority-rule social choice.[6]

As is apparent in figure 5.5, the iteration of this process is far from random, as evident in the complex patterns that emerge.[7] Just as in the case of

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The University of Michigan Press, 1996

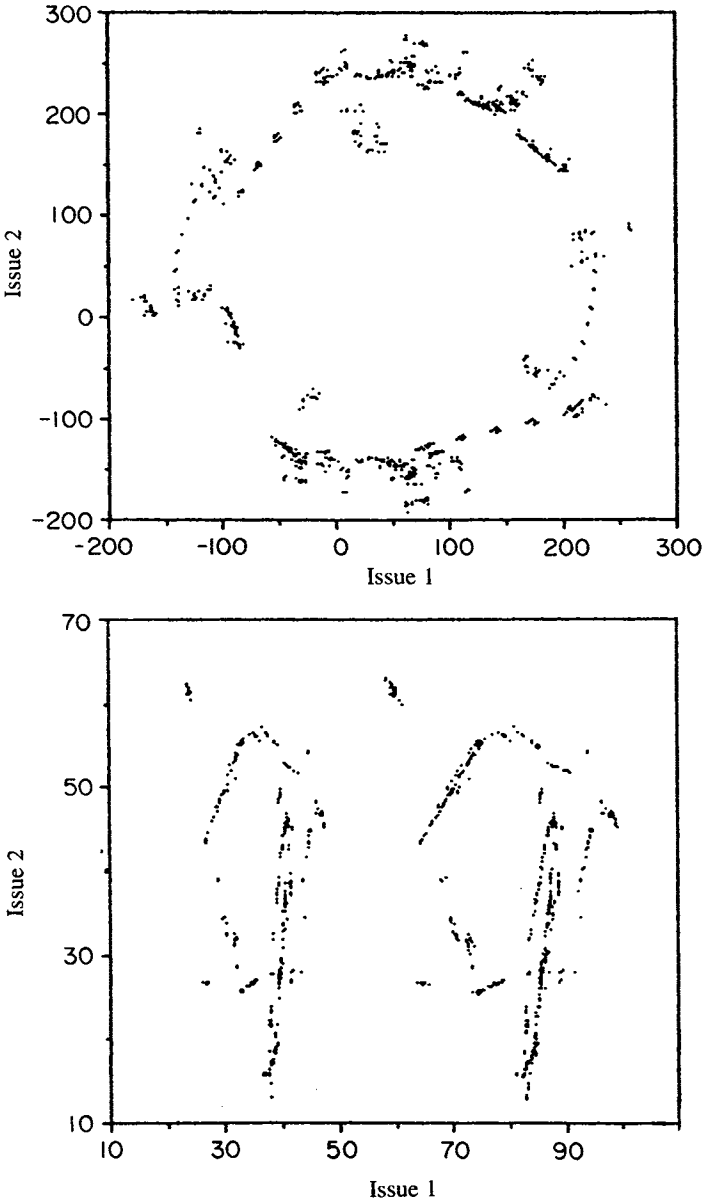

Fig. 5.5. Two samples of the distribution of group choices

the logistic function, there is an order to the dynamic of the majority-rule social choice function.

This order has been anticipated by previous studies. For example, there has been a proliferation of new solution concepts that seek to capture some of the order to voting outcomes. As a sampling, some of the best known are the generalized median set, the minmax set (Kramer 1977), and the uncovered set (McKelvey 1986). It is not necessary to review or define any of these concepts, but they do have some relevant common properties. First, these sets are typically constructed as descriptions of likely outcomes. For example, Kramer (1977) notes that over time, sequences roughly converge onto the minmax set. Ferejohn et al. (1980, 1984) suggest a probabilistic convergence of outcomes toward the generalized median set. Second, these sets are all "centrally located"—i.e., they tend to be centered around a region in the geometric center of the voters' ideal points. For example, Feld et al. (1985) demonstrate that agendas that move toward the generalized median set are easier to construct than agendas moving away from the center of the voters. Third, they are all compact solution sets. For example, the minmax set is a polygon bounded by voter median lines. The generalized median set and the uncovered set are circles of a particular radius.

The order suggested by the theory of chaotic dynamics has also been noticed by experimentalists (e.g., Fiorina and Plott 1978; Wilson 1986). They find that outcomes are not only not scattered over the policy space, but tend to be clustered in patterns. For example, Wilson (1986, 404) concludes that "one point is clear; the outcomes selected here are not scattered throughout the alternative space." Rather than covering the entire policy space, most experimental studies find that outcomes cluster with some deviation near the center of the voters' ideal points. I suspect that the experimentalists are capturing portions of the structure of the chaotic dynamics in their results.

What is happening in the dynamics of social choice is that the individual-level preferences in a context of interaction (i.e., multidimensional majority rule) are creating their own structure, which in turn constrains feasible outcomes. Although there are no external constraints on individuals' votes (in terms of preference orderings, etc.), the voters' actions combine to create a system where each voter, in effect, constrains the other. Game theorists are familiar with this intuition in the important role that the rules of the game have on structuring feasible outcomes (see, e.g., Schelling 1978). Even if individuals have complete autonomy over their actions, the context of their interaction creates structural limits on what individuals can do and therefore on the group outcomes.

Note the parallels between the order implicit in the chaotic dynamics of the individual-group connection and the concept of structure in social science. In the social sciences, structure is something generated by the interaction of

unit-level behavior, yet conditions and constrains system-level options. However, structure in social science is not an external, concrete entity (as an institution might be). "Structure is not something we see. . . . [S]ince structure is an abstraction, it cannot be defined by enumerating material characteristics of the system. It must instead be defined by the arrangement of the system's parts and by the principle of that arrangement" (Waltz 1979, 80). "Structure is a system-wide component that makes it possible to think of the system as a whole" (Waltz 1979, 79). It is this structure that makes a chaotic system *indecomposable*. The whole is complex (disorder), but it is united by a common endogenous structure (order), whereby the system cannot be broken down into additive subsystems.

Chaotic systems are disorderly. They are characterized by an acute sensitive dependence on initial conditions, where slight changes in individual-level input lead to radically different social outcomes. This characteristic in turn leads to an impossibility of long-term prediction based on empirical observation. The dynamics appear random and, no matter how long the dynamic is observed, no repetitious, deterministic patterns will be evident. Prediction must shift from point predictions to boundary predictions, from long-run to short-run. Chaotic dynamics are also disorderly because of the vast diversity of achievable outcomes. Outcomes are not limited to a few, often-reached, repeated outcomes. Any outcome can be reached, from somewhere, via some route, by some length of time. As others have already established, group outcomes are possible anywhere in the policy space.

But chaotic systems are also orderly. They incorporate rigid and ultimately describable constraints on mappings and on routes from group choice to group choice. Previously made choices do constrain future options. Paths are closed, but paths are also continually opened. Social choices are not completely unrelated to individual preferences; there are constraints, but these constraints are "strange." Chaotic systems are also orderly in that they display a fine structure in the distribution of points and the paths. I have alluded to this complex structure in the patterns of the bifurcation diagrams and of the distribution of outcomes from the dynamic. But the structure is multifaceted. It is not merely a circle or a polygon. It is a unique endogenous structure created by the aggregation of individual-level behavior in a given context. Chaotic systems are also orderly in that there is a "consistency of weirdness": stable regions adjoin stable regions and chaotic ones are near chaotic ones. There is a smoothness in the dynamical profile of the social choice function, as made clear by the condition of dense periodic points.

We have long known that the aggregation of individual choices is problematic. If we can understand this mapping as a chaotic process then we have some concepts and tools available that allow for the incorporation of a mix of order and disorder. The process from individual to group is not a regular,

predictable, determinate process. Yet neither is it an incoherent, completely random process. It is this blend that is so unique—both to chaotic dynamics as a general framework and to the aggregation of individual preferences in particular. The individual-group connection has order and constraints that it creates itself as a result of the dynamics of the social choice function. However, the individual-group connection is also capable of infinite variety and a perpetual evolutionary dynamic, where—*even assuming deterministic individual behavior rules*—group outcomes cannot be predicted in the long run.

Chaos, Social Choice, and Democracy

The presence of instability in social choice has already been discussed in terms of the implications for democracy. These implications fall into two categories: (1) the view that the instability makes certain forms of democracy, such as populism, infeasible (Riker 1982), and (2) the view that the instability may help the political stability of pluralist democracy (Miller 1983). I will briefly outline some of the implications for each of these views in light of the individual-group connection being a chaotic process, although I will not attempt to resolve the implicit normative debate.

View 1: Incoherent Social Choices

Given a social choice function that results in outcomes that can be anywhere and that are not consistent with individual preferences, Riker (1982) makes the case that liberal democracy, rather than democracy in its populist form, is more effective. Liberal democracy does not attempt to represent its citizens' preferences, but is democratic in that citizens have the opportunity to remove their representatives. Populism, on the other hand, seeks to aggregate citizens' preferences on specific issues, achieving a social choice by combining and weighing citizen's views. However, based on the pervasive presence of instability in the social choice function, Riker asserts that liberalism is the only democracy that works, since it bypasses the instability problems of the social choice function. Populist democracy, being dependent on an imperfect and troublesome aggregation scheme, will only result in arbitrary decisions and continually changing social outcomes.

What if these aggregation schemes, including the individual preference-group choice connection, are chaotic? The instability was previously known. The new information is that the aggregation function that populism relies on is a chaotic process. Returning to the technical characteristics leads to several implications.

First, group outcomes will appear arbitrary. Studying the relationship

between outcomes over time without knowing the mapping rule, the choice will appear random. No amount of empirical observation will allow for prediction. However, as seen in both examples, these outcomes are far from random. Outcomes are strictly related to individual preferences. They are not arbitrary, in the sense that the same set of individual preferences results in different outcomes at different times. But outcomes are very sensitive to slight changes in these preferences and the connection between individual and group is complex. Complex does not mean random or arbitrary, however. Outcomes in the logistic function were complex, but they were quite "regular" in terms of resulting from a strict deterministic mapping.

Although the mapping is certainly not random or arbitrary, this may not be enough for populism to work. Perhaps the very presence of any instability must be removed. If we understand that the aggregation function is chaotic, what does this tell us about removing the instability? First, it says one can't solve the problem by breaking it down into smaller problems. If there is instability in the whole group, trying to break it down into stable subsystems won't work. This tends to imply that if population won't work in its purest form then neither will a type of "hierarchical populism." Chaotic systems are indecomposable.

Second, it tells us why the instability is occurring. It is not merely unstable; it is unstable because it is in the chaotic regime, which in turn implies that it is nonlinear and there is some "functional folding" or feedback occurring. As seen in both the logistic and spatial voting, it is this feedback process that creates the instability. In theory, if one could "unfold" the social choice function then the instability would disappear![8] By understanding the aggregation process as chaotic, one can examine why it is occurring. If one knows why the mapping is chaotic, then one can systematically, rather than haphazardly, outline ways to reduce the instability, rather than identifying isolated cases where instability does and does not occur.

View 2: Instability Fosters Political Stability

The case for the instability of social choice allowing for a context that supports the stability of pluralism is made by Miller (1983). He reasons that the instability of the social choice process in fact helps promote pluralism. Pluralism views preferences as largely determined by the issue group to which an individual belongs, with society divided along one or more lines that partition the individuals into sets. Where the arrangement is pluralistic, these cleavages are in a crosscutting rather than a reinforcing pattern: individuals that disagree on one issue are likely to agree on other issues. In addition, intensity over issues is also dispersed. The argument of pluralism is that pluralistic prefer-

ences contribute to the stability of the political system. Miller makes the argument that the instability of the social choice function contributes to the stability of pluralism. As Miller (1983, 744) states: "not only does each competitor 'win some and lose some,' but most wins and losses are themselves reversible. Thus the competitors can never be confident of their victories, nor need they resign themselves to their defeats." It is this indeterminacy and instability of winners created by the aggregation process that induces losers "to continue to play the political game, to continue to work within the system rather than try to overthrow it" (1983, 744). Pluralism only guarantees that different people will win or lose on different issues. The instability of the social choice function guarantees that losers on a given issue can hold out in hope of being winners on the same issue.

The presence of an instability in the social choice mapping may promote the stability of a democratic system, but it does not imply any sort of efficiency. As Rae (1982) points out in his metaphor of disequilibrium, democratic decision making becomes an "Escher staircase": leading always up but only coming back to its own foundation. But this instability may be too costly to justify any advantages in promoting political stability if the outcomes are completely arbitrary and unrelated to individual preferences (Miller 1983, 744). Miller did not resolve this issue, but suggested (1983, 745) that "most of us would view political outcomes in the real world of pluralism as considerably unpredictable but clearly confined within certain bounds of 'political feasibility.'" A chaotic aggregation process would certainly support Miller's conjecture. A social choice mapping in the chaotic domain would be empirically unpredictable. This is one of the characteristics of chaos and it arises from the sensitive dependence on initial conditions. Second, the chaotic function also does contain very definite bounds and restrictions—this is the order in the complexity of outcomes. Outcome and paths are highly constrained, although the variety of outcomes is endless, as evident in both the logistic functions and the spatial voting example. The topological transitivity of chaos gives a variety of outcomes that one wouldn't otherwise get without changing preferences. Finally, outcomes are not arbitrary or completely unrelated to public opinion. Just like the logistic function, preferences map to outcomes in a strict deterministic fashion. However, in a chaotic process, one gets an indeterminacy from a determinate mapping (Huckfeldt 1990). A deterministic mapping (majority rule) from individual to group leads to indeterminate, complex, evolutionary outcomes. This is quite unusual. Even if one assumes deterministic choice and behavior rules on the part of individuals, one gets an indeterminacy of social outcomes! By my interpretation of Miller, this is the best combination for democratic political stability: constraints, bounds, and a nonarbitrariness in the individual-group mapping, combined with an unpredictability and evolutionary variety of outcomes over time.

Conclusion

The purpose of this chapter was to reraise the question of the "black box" of the individual-group connection. We know that this connection is not the simple additive relation that early theorists once assumed. We know it is "strange," as evidenced by complex outcomes, instability, and sensitivity. In this chapter I sought to demonstrate that this black box of aggregation from individual to group is in the realm of chaos theory. The function that maps preferences to outcomes and that maps outcomes to outcomes through democratic voting is chaotic—i.e., a unique type of mapping with a blend of disorder—in terms of sensitivity, unpredictability, and evolutionary instability—and of order—in terms of strict mapping rules, regularity in regions of instability, and bounds or constraints on outcomes and paths. I showed this with the example of multidimensional spatial voting: the voting context with the most pervasive presence of instability. However, I suspect this applies to most aggregations, as I suggested by pointing to similar findings by Saari on voting and by economists in terms of aggregation through price mechanisms.

The normative question of whether this is good or bad for liberalism, populism, pluralism, or democracy in general remains open. But it suggests several future tasks that need to be completed. I have shown that multidimensional spatial voting is a chaotic process. This only sets up the problem of understanding what is causing the chaos. Much work needs to be done on why the social choice function is chaotic. Also, much work remains on specifying the general conditions of instability and ways to reduce the instability. But if we know what type of problem we are dealing with, then we can begin to systematically understand the origins of the instability. Only then can we begin to open the black box of the aggregation functions and to resolve the normative debates in democratic theory.

NOTES

Epigraph from *The Social Contract and Discourse*, Book 2, chapter 3. Quoted in Frolich and Oppenheimer 1978,15.

1. Obviously Bentham's scheme raises problems of interpersonal comparison of utility. This is a related issue of such magnitude that it will not be discussed here.

2. Occasionally I will use the term *social choice function* rather than *social choice correspondence* nontechnically, as it is used throughout the voting literature, to designate a mapping from individual preferences to a social choice.

3. Sarkovskii's theorem (that a three-cycle implies the existence of cycles of all lengths) applies to the logistic function since it is one-to-one and continuous. These conditions do not hold for the social choice correspondence.

4. *Completeness* requires that all alternatives can be ranked.

5. However, Saari (1990) demonstrates that the Borda count reduces the number of paradoxes.

6. Of course, this distribution is only one of many potential examples, but it illustrates the presence of order.

7. The connection with fractals is suggested by the complex patterns in the distribution of outcomes.

8. This is the subject of a future research project.